Isometries of combinatorial Tsirelson spaces

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IWOTA 2022

Outline

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Introduction

- combinatorial Tsirelson spaces;

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- Tsirelson-type spaces;
- Tsirelson spaces (the first example of spaces containing no isomorphic copies of c_0 or ℓ_p for $1 \leq p < \infty$).

Introduction

What are the (linear) isometries of combinatorial Tsirelson spaces?

Basic concepts

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- For $n \in \mathbb{N}$
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- For $x_1, x_2 \in c_{00}$
$$x_1 < x_2 \text{ if } \max \operatorname{supp} x_1 < \min \operatorname{supp} x_2.$$

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Definition

A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is **regular**, whenever it is simultaneously

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- **spreading** $\big(\{l_1, l_2, \ldots, l_n\} \in \mathcal{F}$ and $l_i \leqslant k_i \implies \{k_1, k_2, \ldots, k_n\} \in \mathcal{F}\big)$;

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- **compact** as a subset of the Cantor set $\{0,1\}^{\mathbb{N}}$ via the natural identification of $F \in \mathcal{F}$ with

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Example

$$\mathcal{A}_n := \big\{F \in [\mathbb{N}]^{<\omega} \colon |F| \leqslant n\big\} \cup \big\{\emptyset\big\} \quad (n \in \mathbb{N})$$

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$\mathcal{S}_2$-sets

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- $\{2,3,4,7,100,5\} = \{2,3\} \cup \{4,7,100,5\}$

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- $\{2,3,4\} = \{2,3\} \cup \{4\}$
- $\{2,3,4,7,100,5\} = \{2,3\} \cup \{4,7,100,5\}$
- $\{3,5,7,n,n+1,\ldots,2n-1\} = \{3,5\} \cup \{7\} \cup \{n,n+1,\ldots,2n-1\}$

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- if α is **a successor ordinal**

$$\mathcal{S}_{\alpha+1} := \left\{ \bigcup_{i=1}^{d} S_\alpha^i \colon d \leq S_\alpha^1 < S_\alpha^2 < \cdots < S_\alpha^d \text{ and } \left\{ S_\alpha^i \right\}_{i=1}^{d} \subset \mathcal{S}_\alpha \right\} \cup \{\emptyset\};$$

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- if α is **a limit ordinal** and $(\alpha_n)_{n=1}^{\infty}$ is a fixed strictly increasing sequence of ordinals converging to α

$$\mathcal{S}_\alpha := \left\{ S_{\alpha_n} \in [\mathbb{N}]^{<\omega} \colon S_{\alpha_n} \in \mathcal{S}_{\alpha_n} \text{ for some } n \leq \min S_{\alpha_n} \right\} \cup \{\emptyset\}.$$

Basic concepts

Notation

For $x \in c_{00}$ and $E \in [\mathbb{N}]^{<\omega}$ let ***Ex* be the projection** of the vector x onto the coordinates belonging to E, i.e.

$$E\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i \in E} a_i x_i.$$

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Definition

Fix $\theta \in (0,1)$. Let $\mathcal{F}$ be regular family and $\|\cdot\|_0$ **be the supremum norm** on c_{00}. Suppose that for some $n \in \mathbb{N}$ the norm $\|\cdot\|_n$ **has been defined.**

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$$\|\boldsymbol{x}\|_{\boldsymbol{n+1}} := \max\Bigg\{\|x\|_n, \\ \sup\Big\{\theta \sum_{i=1}^{d} \big\|E_i x\big\|_n : E_1 < \cdots < E_d \text{ in } [\mathbb{N}]^{<\omega} \text{ and } \{\min E_i\}_{i=1}^{d} \in \mathcal{F}\Big\}\Bigg\}$$

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and denote by $T[\theta,\mathcal{F}]$ the completion of c_{00} with respect to it.

Basic concepts

Examples

- $T\left[\theta, \mathcal{S}_\alpha\right]$ for $\theta \in (0,1)$, $1 \leqslant \alpha < \omega_1$ - combinatorial Tsirelson spaces;
- $T\left[\theta, \mathcal{S}_1\right]$ for $\theta \in (0,1)$ - Tsirelson-type spaces;
- $T\left[\frac{1}{2}, \mathcal{S}_1\right]$ - Tsirelson spaces.

Motivations

Theorem (K. Beanland)

Let $n \in \mathbb{N}$ with $n \geq 2$. Then $U\colon T\left[\frac{1}{n}, \mathcal{S}_1\right] \to T\left[\frac{1}{n}, \mathcal{S}_1\right]$ **is an isometry iff**

$$Ue_i = \begin{cases} \pm e_{\pi(i)}, & 1 \leq i \leq n \\ \pm e_i, & i > n \end{cases} \qquad (i \in \mathbb{N})$$

for some permutation π of $\{1, 2, \ldots, n\}$.

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Problem

Characterize surjective isometries on $T[\theta, \mathcal{F}]$ for $\theta \in (0,1)$ and regular families $\mathcal{F}$. The case of $\theta = \frac{1}{2}$ and $\mathcal{S}_\alpha$ for some countable ordinal $\alpha > 1$ is especially interesting.

Main theorem

Theorem 1

Let $\theta \in \left(0, \frac{1}{2}\right]$. If $U\colon T\left[\theta, \mathcal{S}_1\right] \to T\left[\theta, \mathcal{S}_1\right]$ **is an isometry,** then

$$Ue_i = \begin{cases} \pm e_{\pi(i)}, & 1 \leq i \leq \lceil \theta^{-1} \rceil \\ \pm e_i, & i > \lceil \theta^{-1} \rceil \end{cases} \qquad (i \in \mathbb{N})$$

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Theorem 2

Let $\theta \in \left(0, \frac{1}{2}\right], 1 < \alpha < \omega_1$. Then an operator $U\colon T[\theta, \mathcal{S}_\alpha] \to T[\theta, \mathcal{S}_\alpha]$ **is an isometry iff** $Ue_i = \pm e_i$ for $i \in \mathbb{N}$.

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The idea of the proof of Th. 2. for $\mathcal{S}_2$-sets

For any $m \in \mathbb{N}$ find

$$y_1 < y_2 < \cdots < y_m$$

such that

$$\|y_i - x_{j_i}\| < \varepsilon, \quad i = 1, 2, \ldots, m$$

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- $(e_j)_{j=1}^{\infty}$ is weakly null,
- c_{00} is dense in $T[\theta, \mathcal{S}_2]$.

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Repeat it ('j_1- many' times) to obtain a maximal $\mathcal{S}_2$-set from indieces j_i.

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Thank you for your attention!

Bibliography

N. Maślany
Isometries of combinatorial Tsirelson spaces
arXiv:2209.00113 [math.FA], 2022

L. Antunes, K. Beanland
Surjective isometries on Banach sequence spaces: A survey
Concrete Operators, 2022

C. Brech, V. Ferenczi and A. Tcaciuc
Isometries of combinatorial Banach spaces
Proc. Am. Math. Soc., 2020

B. Tsirelson
It is impossible to embed ℓ_p or c_0 into an arbitrary Banach space (Russian)
Funkts. Anal. i Prilozhen English translation: Funct. Anal. Appl., 1974